Resonance Instability for Finned Configurations Having Nonlinear Aerodynamic Properties

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An approximate solution is presented for the angular motion, at and near resonance, of finned configurations (with small aerodynamic asymmetries) having nonlinear aerodynamic properties. An extension of the method of slowly varying parameters is used to solve the simultaneous pitch and yaw equations of motion, in terms of amplitude-rate equations, at the resonance point. Included in this analysis are the nonlinear variations of the restoring, damping, and Magnus moments with angle of attack. Both transient and steady-state solutions are obtained. A detuning of the rolling velocity from its resonant value is performed to yield a solution in the near-resonant region. This solution is shown to give rise to the well-known resonance jump in the steady-state case. Finally, allowing the detuning parameter to vary with time enables an examination of the nonlinear catastrophic yaw problem in which both the rolling velocity and roll orientation angle may vary with time. In this regard, the effects of a nonlinear variation of the induced side moment with angle of attack are determined. The accuracy of these solutions is evaluated by comparison with exact numerical integration of the equations of motion.

Nomenclature

d	= diameter
I,I_X	= transverse and axial moments of inertia
$K_{N,P,T}$	= arms of tricyclic theory
$\mathbf{K}_{N,P,T}$	= arms of nonlinear theory
L,M,N	= roll, pitch, and yaw moments
M_α	= restoring moment stability derivative, $\partial M/\partial\alpha$
M_q	= damping moment stability derivative, $\partial M/\partial(qd/2V)$
$M_{\dot\alpha}$	= lag moment stability derivative, $\partial M/\partial(\dot\alpha d/2V)$
$M_{p\alpha}$	= Magnus moment stability derivative, $\partial^2 N/\partial\alpha\partial(pd/2V)$
$M(\gamma,\boldsymbol{\alpha})$	= induced side moment
$\mathbf{M}(\delta\epsilon)$	= complex asymmetry moment
p,q,r	= angular velocity components
p_r	= rolling velocity at resonance
s	= gyroscopic stability factor
t	= time
V	= total velocity
$\boldsymbol{\alpha}$	= complex angle of attack, $\beta + i\alpha$
γ	= roll orientation angle
Γ	= nondimensional restoring moment nonlinearity, $M_{\alpha_2}\lvert\boldsymbol{\alpha}\rvert ss^2/M_{\alpha_0}$
λ	= damping factor
ω	= frequency

Subscripts

0	= linear term
2	= second-order, nonlinear term
N,P	= nutation and precession, respectively

Introduction

BECAUSE of configurational asymmetries, finned vehicles may experience large angular motions when the roll rate is near the nutation frequency. The increase in amplitude of total yaw is characterized by an amplification of the nonrolling trim angle of attack: resonance instability.[1,2] This phenomenon and the related roll-induced catastrophic yaw[3] have been recognized and analyzed under the assumptions of linear or quasi-linear[2] aerodynamics. It is necessary, therefore, because of the large angles characteristic of these phenomena, to examine the dynamic stability of finned vehicles at resonance when nonlinear aerodynamic properties are present.

The effects of angle-of-attack nonlinearities on the angular motion of missiles has been extensively pursued.[4-7] In addition, the effects of configurational asymmetries have been determined in the presence of these nonlinearities, but not at resonance.[8,9] With regard to the resonance problem, only steady-state solutions in the presence of a nonlinear restoring moment have been discussed.[10] It is the purpose of this paper to present an approximate solution for both the transient and steady-state angular motions of a finned vehicle possessing aerodynamic properties nonlinear in angle of attack at and near resonance. The subject matter for this paper is taken from Ref. 11, which includes considerable detail with regard to obtaining and evaluating the nonlinear solutions.

In the previous nonlinear analyses,[4-10] the equations of motion have been used in their convenient complex form because of the symmetry considerations.[12] The approach of this paper applies the method of slowly varying parameters[13-15] to the simultaneous pitch and yaw equations of motion, which removes the symmetry restriction, although a symmetric configuration is assumed for convenience. The resulting amplitude-rate equations provide a system of first-order differential equations in the amplitudes (and phases) of the assumed solution. Although, in the most general cases, these equations must be solved numerically, they reduce the problem from one involving second order differential equations in the angles of attack and sideslip to a system of first order equations in the amplitudes of oscillation. Furthermore, equations of this nature are well suited for stability analyses, the prediction of possible limit cycles and their amplitudes, etc.

Linear Theory

Linear aeroballistic theory[2] concerns itself with obtaining a solution for the angular motion of a missile with linear aerodynamic properties at a constant value of the rolling velocity.

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The complex differential equation of motion (in aeroballistic axes[2]) and tricyclic solution are summarized below:

$$\ddot{\boldsymbol{\alpha}} - [ipI_X/I + (M_q + M_{\dot{\alpha}})/I]\dot{\boldsymbol{\alpha}} - (M_\alpha/I + ipM_{p\alpha}/I)\boldsymbol{\alpha} = [i\mathbf{M}(\delta_\epsilon)/I]\exp(ipt) \quad (1)$$

$$\boldsymbol{\alpha} = \mathbf{K}_N \exp(\lambda_N + i\omega_N)t + \mathbf{K}_P \exp(\lambda_P + i\omega_P)t + \mathbf{K}_T \exp(ipt) \quad (2)$$

$$\lambda_{N,P} = (M_q + M_{\dot{\alpha}})(1 \pm \tau)/2I \pm M_{p\alpha}\tau/I_X \quad (3)$$

$$\omega_{N,P} = (pI_X/2I)(1 \pm \tau^{-1}) \quad (4)$$

$$\tau = 1/(1 - 1/s)^{1/2} \qquad s = (pI_X)^2/4IM_\alpha \quad (5)$$

The amplification of the nonrolling trim is given by

$$|\mathbf{K}_T/\mathbf{K}_T|_{p=0}| = [(-M_\alpha/I)^2/(h_1^2 + h_2^2)]^{1/2} \quad (6)$$

where

$$h_1 = \omega_P(p - \omega_N) + p(\omega_N - p)$$
$$h_2 = \lambda_P(\omega_N - p) + \lambda_N(\omega_P - p)$$

Equation (6) is seen to be a maximum when $p \approx \omega_N$. The inclusion of the induced side moment, $M(\gamma,\boldsymbol{\alpha})$, when considering catastrophic yaw, modifies the damping factors $\lambda_{N,P}$.[3]

Nonlinear Theory

The nonlinear forced oscillation problem has been extensively pursued for single-degree-of-freedom systems[16] from which several phenomena associated with nonlinear resonance have been obtained; e.g., the amplification of the forcing function is most sensitive to the degree of non-linearity of the spring constant (or restoring moment).

Because of the nonlinear nature of the problem, the term resonance becomes somewhat obscure. For the present analysis, resonance is defined as that point at which the rolling velocity equals the linear natural frequency, ω_{N_0}. Therefore, from Eq. (4),

$$p_r = \omega_{N_0} = (p_r I_X/2I)(1 + \tau^{-1}) \quad (7)$$

where τ is computed using the linear part of the nonlinear restoring moment.

The nonlinear differential equations of motion may be written

$$\begin{aligned}\ddot{\alpha} - (pI_X/2I)\dot{\beta} - (M_{\alpha_0}/I)\alpha &= [M_{\alpha_2}/I](\alpha^2 + \beta^2)\alpha + [(M_q + M_{\dot{\alpha}})_0/I]\dot{\alpha} + [(M_q + M_{\dot{\alpha}})_2/I](\alpha^2 + \beta^2)\dot{\alpha} + [\{M_{p\alpha_0}p + M_\alpha(\gamma)_0\}/I]\beta + [\{M_{p\alpha_2}p + M_\alpha(\gamma)_2\}/I](\alpha^2 + \beta^2)\beta + [M(\delta_\epsilon)/I]\cos pt \\ \ddot{\beta} + (pI_X/2I)\dot{\alpha} - (M_{\alpha_0}/I)\beta &= [M_{\alpha_2}/I](\alpha^2 + \beta^2)\beta + [(M_q + M_{\dot{\alpha}})_0/I]\dot{\beta} + [(M_q + M_{\dot{\alpha}})_2/I](\alpha^2 + \beta^2)\dot{\beta} - \{[M_{p\alpha_0}p + M_\alpha(\gamma)_0]/I\}\alpha - \{[M_{p\alpha_2}p + M_\alpha(\gamma)_2]/I\}(\alpha^2 + \beta^2)\alpha - [M(\delta_\epsilon)/I]\sin pt\end{aligned} \quad (8)$$

where nonlinearities up to third order in angle of attack have been considered. $M_\alpha(\gamma)_{0,2}$ represent the linear and nonlinear components of the induced side moment stability derivative.

Angular Motions at Resonance

For this analysis, p is assumed constant, as is $M_\alpha(\gamma)_{0,2}$, which implies γ is constant (equal to, say, the roll trim angle at lock-in). While the latter is not physically realistic for extended time periods, the induced side moment is included for completeness, but will not be considered in the solution at resonance. It will be considered, however, in the nonlinear catastrophic yaw analysis.

Before solving Eqs. (8), it is convenient to solve first two coupled, second-order, nonlinear differential equations in general and apply the results to Eqs. (8).

Several approaches have been formulated for analyzing nonlinear multi-degree-of-freedom systems.[17–19] The resonance problem has also been analyzed.[20,21] It was found that an extension of the method of slowly varying parameters[13] was most applicable to the problem at hand. This method, as applied to a single nonlinear differential equation, consists of the familiar procedure of assuming a solution, and first derivative, of the form of the linearized equation, and substituting it back into the original nonlinear equation (assuming slowly varying amplitude and phase) to arrive at first-order differential equations in both parameters. The same is applied here, extended to a pair of equations. Consider the gyroscopically coupled system

$$\begin{aligned}\ddot{x} - P\dot{y} + (\pm n_1^2)x &= \mu f(x,\dot{x},y,\dot{y}) \\ \ddot{y} + P\dot{x} + (\pm n_2^2)y &= \mu g(x,\dot{x},y,\dot{y})\end{aligned} \quad (9)$$

where μ is the associated smallness parameter.[22] The solution to the linearized equations ($\mu = 0$) may be easily found to be

$$\begin{aligned}x &= a\sin(k_1t + \delta_1) + b\sin(k_2t + \delta_2) \\ y &= \sigma_1 a\cos(k_1t + \delta_1) + \sigma_2 b\cos(k_2t + \delta_2)\end{aligned} \quad (10)$$

where

$$\begin{aligned}k^4 + [(\mp n_1^2) + (\mp n_2^2) - P^2]k^2 + n_1^2n_2^2 &= 0 \\ \sigma_{1,2} = [(\mp n_1^2) + k_{1,2}^2]/Pk_{1,2} &= Pk_{1,2}/[\mp n_2^2 + k_{1,2}^2]\end{aligned} \quad (11)$$

The parameters a, b, δ_1, δ_2 are functions of the initial conditions. Application of the method of slowly varying parameters leads to the following expressions for the rates of change of amplitudes and phases[11]:

$$\begin{aligned}\dot{a} &= \psi[F_1/\sigma_1 - G_1] & \dot{b} &= \psi[-F_2/\sigma_2 + G_2] \\ a\dot{\delta}_1 &= -\psi[F_3/\sigma_1 - G_3] & b\dot{\delta}_2 &= \psi[F_4/\sigma_2 + G_4]\end{aligned} \quad (12)$$

where

$$\begin{bmatrix}F_1\\F_2\\F_3\\F_4\end{bmatrix} = 1/(4\pi^2)\int_0^{2\pi}\int_0^{2\pi} f^* \begin{bmatrix}\cos\xi\\ \cos\eta\\ \sin\xi\\ \sin\eta\end{bmatrix} d\xi d\eta$$

$$\begin{bmatrix}G_1\\G_2\\G_3\\G_4\end{bmatrix} = 1/(4\pi^2)\int_0^{2\pi}\int_0^{2\pi} g^* \begin{bmatrix}\sin\xi\\ \sin\eta\\ \cos\xi\\ \cos\eta\end{bmatrix} d\xi d\eta$$

$$\psi = \mu P/(k_1^2 - k_2^2) \qquad \xi = k_1t + \delta_1 \qquad \eta = k_2t + \delta_2$$

The functions f^* and g^* are f and g evaluated at x, $\dot{x}$, y, $\dot{y}$ of the generating solution.

Although the above approximate solution refers to an autonomous system, it serves as a basis for consideration of the nonautonomous, resonance system. Unfortunately, as pointed out by many authors,[13–15] because of the nonlinear nature of the problem, resonance may not be treated as a special case of general, nonautonomous systems but must be considered separately.

It is well-known that, far from resonance, the amplification of the forcing function magnitude is relatively insensitive to the damping in the system. Hence, with regard to the nonautonomous, nonresonant situation, the generalized problem [as in Eq. (9)] may be formulated by simply adding the forcing term directly to the right-hand side of Eq. (9). Consequently, the generating solution ($\mu = 0$) will consist of Eq. (10) plus a particular solution either 0° or 180° out of phase with one of the natural frequencies, depending on whether the forcing is below or above resonance, respectively.

Oscillations at resonance, however, must be handled in a slightly different manner since the amplification is critically dependent on damping. In this regard, since the general formulation lumps the nonconservative terms with the nonlinear terms, it seems reasonable to assume that the external

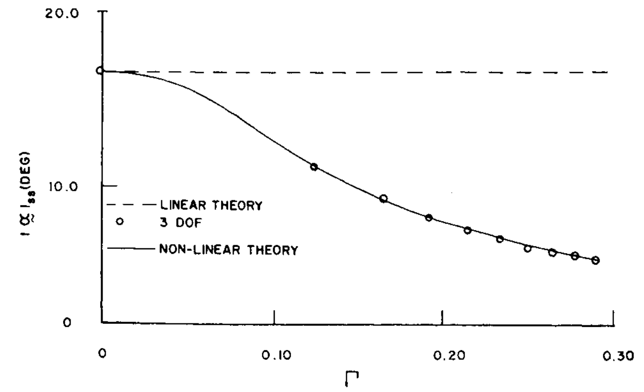

Fig. 1 Steady-state effect of nonlinear restoring moment at resonance.

excitation acts in a similar manner, viz.

$$\begin{aligned} \ddot{x} - P\dot{y} + (\pm n_1^2)x &= \mu f(x,\dot{x},y,\dot{y}) + \mu Q \cos pt \\ \ddot{y} + P\dot{x} + (\pm n_2^2)y &= \mu g(x,\dot{x},y,\dot{y}) - \mu Q \sin pt \end{aligned} \tag{13}$$

Since resonant oscillations $p = k_1$ are under consideration, consider a solution of the form[11,13]

$$\begin{aligned} x &= a_1 \sin pt + a_2 \cos pt + b \sin(k_2 t + \delta_2) \\ y &= \sigma_1 a_1 \cos pt - \sigma_2 a_2 \sin pt + \sigma_2 b \cos(k_2 t + \delta_2) \end{aligned} \tag{14}$$

where the a_1 mode is the response to the external excitation (note the 90° phase shift), the a_2 mode is the natural mode oscillating at k_1 $(= p)$, and the b mode is the other natural mode as before. The remaining equations (frequency, etc.) necessary to define the generating solution are those for the autonomous case: Eqs. (11). The four constants of the generating solution may be found from

$$\begin{aligned} a_1 &= [\dot{x}(0) - y(0)k_2]/(p - k_2) \\ a_2 &= [-\dot{y}(0) - x(0)k_2]/(p - k_2) \\ b &= \{[\dot{y}(0) + px(0)]^2 + [y(0)p - \dot{x}(0)]^2\}^{1/2}/(p - k_2) \\ \delta &= \tan^{-1}\{[\dot{y}(0) + px(0)]/[y(0)p - \dot{x}(0)]\} \end{aligned} \tag{15}$$

Applying the method of slowly varying parameters to the generating solution yields

$$\begin{aligned} p\dot{a}_1 &= \Psi_1[F_1 + Q/2 - \sigma_1(G_1 - Q/2)] \\ p\dot{a}_2 &= -\Psi_1[F_2 + \sigma_1 G_2] \\ k_2\dot{b} &= -\Psi_2[F_3 - \sigma_2 G_3] \qquad b\dot{\delta}_2 = \Psi_2[F_4 + \sigma_2 G_4] \end{aligned} \tag{16}$$

where

$$\begin{aligned} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} &= 1/4\pi^2 \int_0^{2\pi}\int_0^{2\pi} f^* \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ \cos\xi \\ \sin\xi \end{bmatrix} d\xi d\varphi \\ \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} &= 1/4\pi^2 \int_0^{2\pi}\int_0^{2\pi} g^* \begin{bmatrix} \sin\varphi \\ \cos\varphi \\ \sin\xi \\ \cos\xi \end{bmatrix} d\xi d\varphi \\ \Psi_1 &= [p^2 + (\mp n_2^2)]/(p^2 - k_2^2) \\ \Psi_2 &= [k_2^2 + (\mp n_2^2)]/(p^2 - k_2^2) \\ \varphi &= pt \qquad \xi = k_2 t + \delta_2 \end{aligned} \tag{17}$$

The functions f^* and g^* are f and g [Eqs. (13)] evaluated at x, $\dot{x}$, y, $\dot{y}$ of the generating solution [Eqs. (14)]. Eqs. (14–17), then, represent an approximate solution to Eqs. (13) at resonance, $p = k_1$.

Resonant oscillations of a finned missile

Having established the above generalized formulation of the nonlinear resonance problem, it remains only to compare this with the governing equations of motion for a finned mis-

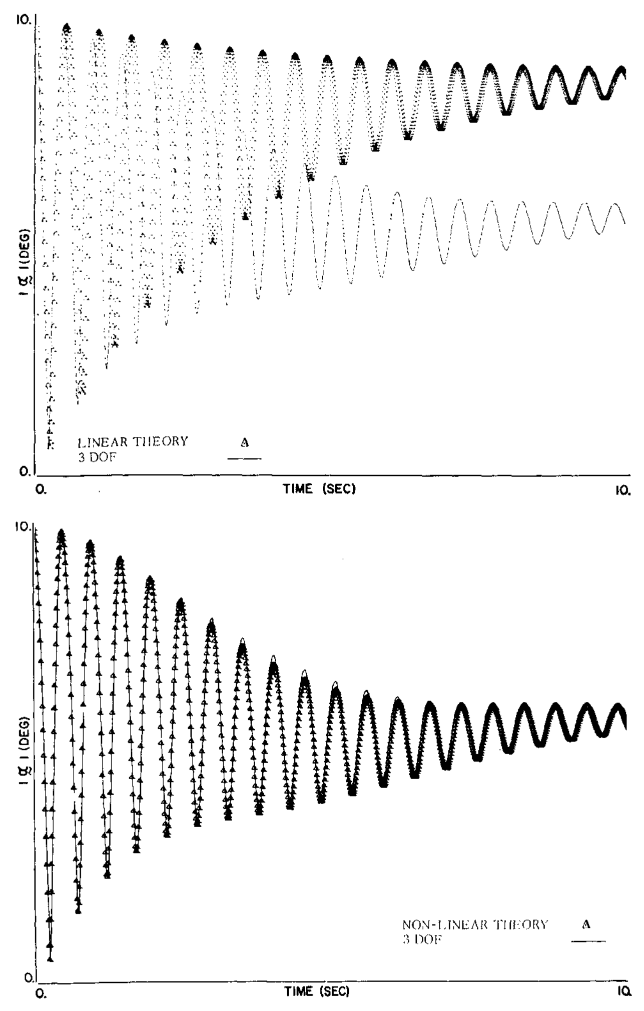

Fig. 2 Comparison of linear (top) and nonlinear (bottom) theories with 3-degree-of-freedom computations for a nonlinear restoring moment.

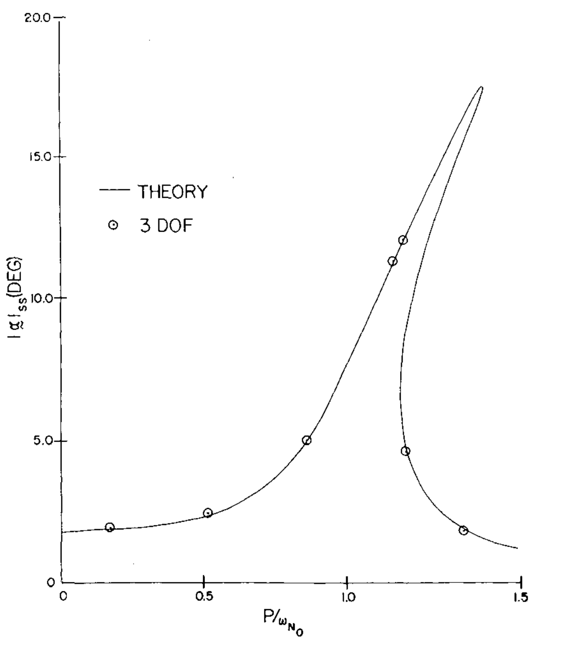

Fig. 3 Steady-state evaluation of near-resonant theory.

sile: Eqs. (8). This leads to

$$x = \alpha \qquad y = \beta \qquad P = p_r I_X / I$$

$$n_1^2 = n_2^2 = -M_{\alpha 0}/I \qquad Q = M(\delta_\epsilon)/I \qquad \sigma_{1,2} = 1$$

Therefore, letting $a_1 = \mathbf{K}_T$, $a_2 = \mathbf{K}_N$, $b = \mathbf{K}_P$, $\delta_2 = \boldsymbol{\delta}_P$ to conform to the notions of nutation, precession, and trim,

$$\begin{aligned} \alpha &= \mathbf{K}_T \sin p_r t + \mathbf{K}_N \cos p_r t + \mathbf{K}_P \sin(\omega_P t + \boldsymbol{\delta}_P) \\ \beta &= \mathbf{K}_T \cos p_r t - \mathbf{K}_N \sin p_r t + \mathbf{K}_P \cos(\omega_P t + \boldsymbol{\delta}_P) \end{aligned} \tag{18}$$

where, since $k_1 = p = p_r$,

$$\begin{aligned} \omega_N = \omega_{N_0} = k_1 = p_r &= (p_r I_X/2I)(1 + \tau^{-1}) \\ \omega_P = k_2 &= (p_r I_X/2I)(1 - \tau^{-1}) \end{aligned} \tag{19}$$

The corresponding rate equations become

$$\dot{\mathbf{K}}_N = A\mathbf{K}_N - B\mathbf{K}_T \tag{20a}$$

$$\dot{\mathbf{K}}_T = A\mathbf{K}_T + B\mathbf{K}_N + M(\delta_\epsilon)\tau/(p_r I_X) \tag{20b}$$

$$\dot{\mathbf{K}}_P = C\mathbf{K}_P \tag{20c}$$

$$\dot{\boldsymbol{\delta}}_P = M_{\alpha_2}(2\mathbf{K}_N^2 + 2\mathbf{K}_T^2 + \mathbf{K}_P^2)\tau/p_r I_X \tag{20d}$$

where

$$\begin{aligned} A = {} & (M_q + M_{\dot\alpha})(1 + \tau)/2I + [M_{p\alpha 0}/I_X + \\ & M_\alpha(\gamma)_0/p_r I_X]\tau + (M_q + M_{\dot\alpha})_2[\mathbf{K}_N^2 + \mathbf{K}_T^2 + \\ & (\mathbf{K}_N^2 + \mathbf{K}_T^2 + 2\mathbf{K}_P^2)\tau]/2I + [M_{p\alpha 2}/I_X + \\ & M_\alpha(\gamma)_2/p_r I_X](\mathbf{K}_N^2 + \mathbf{K}_T^2 + 2\mathbf{K}_P^2)\tau \end{aligned}$$

$$B = M_{\alpha 2}(\mathbf{K}_N^2 + \mathbf{K}_T^2 + 2\mathbf{K}_P^2)\tau/p_r I_X$$

$$\begin{aligned} C = {} & (M_q + M_{\dot\alpha})_0(1 - \tau)/2I - [M_{p\alpha 0}/I_X + \\ & M_\alpha(\gamma)_0/p_r I_X]\tau + (M_q + M_{\dot\alpha})_2[\mathbf{K}_P^2 - \\ & (2\mathbf{K}_N^2 + 2\mathbf{K}_T^2 + \mathbf{K}_P^2)\tau]/2I - \\ & [M_{p\alpha 2}/I_X + M_\alpha(\gamma)_2/p_r I_X](2\mathbf{K}_N^2 + 2\mathbf{K}_T^2 + \mathbf{K}_P^2)\tau \end{aligned}$$

τ is computed using the rolling velocity at resonance, p_r. In addition, from Eq. (15),

$$\begin{aligned} \mathbf{K}_N(0) &= [r(0) - \alpha(0)\omega_P]/(p_r - \omega_P) \\ \mathbf{K}_T(0) &= [q(0) - \beta(0)\omega_P]/(p_r - \omega_P) \\ \boldsymbol{\delta}_P(0) &= \tan^{-1}[-r(0) + p_r\alpha(0)]/[\beta(0)p_r - q(0)] \\ \mathbf{K}_P(0) &= \{[-r(0) + p_r\alpha(0)]^2 + [\beta(0)p_r - q(0)]^2\}^{1/2}/(p_r - \omega_P) \end{aligned} \tag{21}$$

These relations, then, provide the amplitudes and phases of the assumed solution. Although these equations are not, in general, solvable, they do reduce the problem to one in terms of the more readily discernable parameters of amplitude and phase.

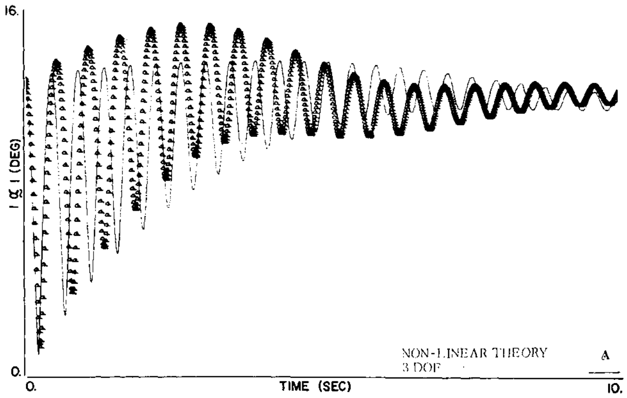

Fig. 4 $|\alpha|$ vs time for the aerodynamics of Fig. 3 at $p/\omega_N = 1.20$.

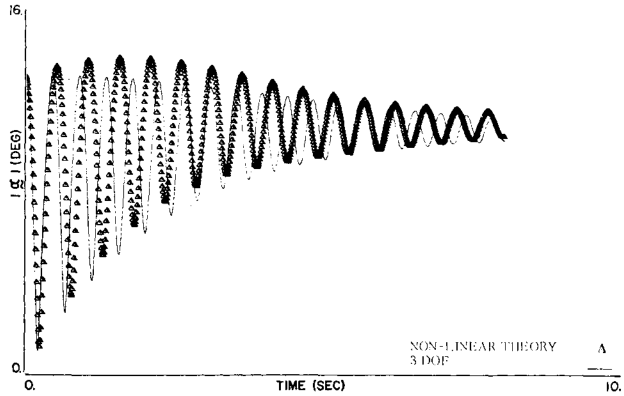

Fig. 5 $|\alpha|$ vs time for nonlinear restoring, damping, and Magnus moments.

Evaluation of nonlinear resonance theory

For the case of linear damping and Magnus moments (neglecting the induced side moment) and a nonlinear restoring moment, Eqs. (20) reduce to

$$\begin{aligned} D^2\mathbf{K}_{T_{SS}}^3 + \mathbf{K}_{T_{SS}} + M(\delta\epsilon)\tau/(\lambda_N p_r I_X) &= 0 \\ \mathbf{K}_{N_{SS}} = -D\mathbf{K}_{T_{SS}}^2 \qquad \mathbf{K}_{P_{SS}} &= 0 \end{aligned} \tag{22}$$

where

$$D = M_{\alpha_2} M(\delta_\epsilon)\tau^2/(\lambda_N p_r I_X)^2$$

The first of Eqs. (22), by nature of the coefficients, possesses only one real root, in agreement with the observation that only one steady-state amplitude exists at the resonance point defined by Eq. (7). Figure 1 presents a comparison of Eqs. (22) with linear theory and three-degree-of-freedom integrations of the Euler differential equations of motion, from which Eqs. (1) were derived. These results are plotted as a function of $\Gamma(= M_{\alpha 2}|\boldsymbol{\alpha}|_{SS}^2/M_{\alpha 0})$, an indication of the nonlinearity of the restoring moment. It is clear that the nonlinear theory accurately predicts the decrease in amplitude with increasing nonlinearity in the restoring moment.

In addition to accurate predictions of steady-state oscillations, the nonlinear theory has been found to represent well the transient angular behavior at resonance. Figure 2 presents a comparison of the linear (top) and non-linear (bottom) theories' predictions of $|\boldsymbol{\alpha}|$ vs time with the 3-degree-of-freedom solution for linear damping and Magnus moments with a hard spring restoring moment. Numerous evaluations of the nonlinear theory have been performed[11] with regard to differing degrees of nonlinearity in the restoring, damping, and Magnus moments and variations of initial conditions (e.g., circular and planar motions). In all cases, good agreement between theory and 3-degree-of-freedom computations was evident.

Angular Motions Near Resonance

The above simulations were constrained to constant rolling velocity at the resonance point ($p_r = \omega_{N_0}$). With regard to near-resonance oscillations, an approximate analytic solution for the angular motion was obtained employing the nonlinear

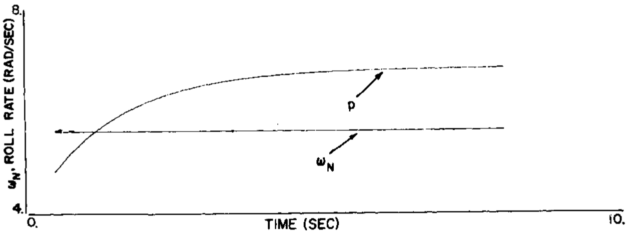

Fig. 6 Roll-rate passage through resonance.

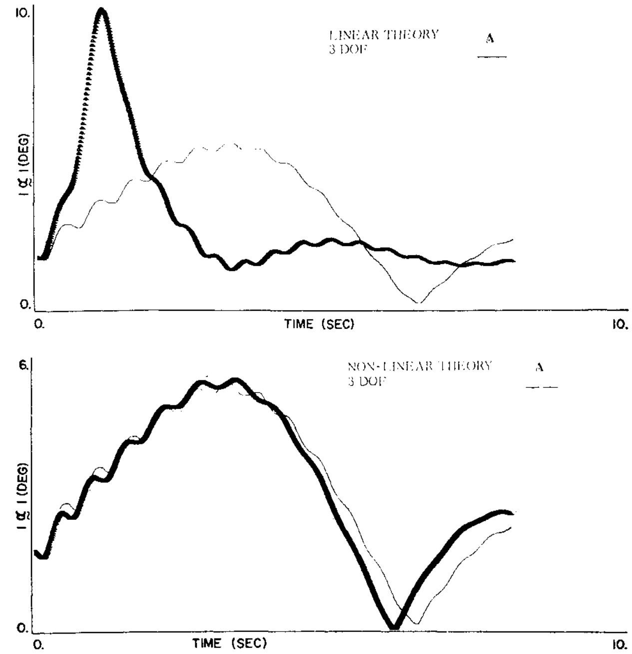

Fig. 7 Comparison of linear (top) and nonlinear (bottom) theories with 3-degree-of-freedom computations for linear aerodynamics.

resonance solution as a base. Consider a detuning of the rolling velocity from its resonant value[15,19]:

$$\omega_{N_0}^2 - p^2 = \mu\Delta \tag{23}$$

where μ is the smallness parameter of the nonlinear theory and Δ is the detuning parameter. If this substitution is made into the nonlinear resonance theory and the detuning term grouped with the nonlinear and forcing terms of Eqs. (13), the resonant solution may still be used as a generating solution. Subsequent application of the method of slowly varying parameters yields a solution near resonance identical to Eqs. (18) and (19) with the related equations

$$\begin{aligned} \dot{\mathbf{K}}_N &= \text{RHS[Eq. (20a)]} + \Delta I(1 - I_X/4I)\tau\mathbf{K}_T/p_r I_X \\ \dot{\mathbf{K}}_T &= \text{RHS[Eq. (20b)]} - \Delta I(1 - I_X/4I)\tau\mathbf{K}_N/p_r I_X \\ \dot{\delta}_P &= \text{RHS[Eq. (20d)]} - \Delta I(1 - I_X E/2I)\tau \\ \dot{\mathbf{K}}_P &= \text{RHS[Eq. (20c)]} \end{aligned} \tag{24}$$

where $E = 1 + (\tau - 1)/(\tau + 1)$.

Evaluation of near-resonance theory

For steady-state oscillations with linear damping and Magnus moments and a nonlinear restoring moment, Eqs. (24) reduce to

$$D^2\mathbf{K}_{T_{SS}}^3 + (2DF\lambda_N^2/p_r)\mathbf{K}_{T_{SS}}^2 + (1 + F^2)\mathbf{K}_{T_{SS}} + M(\delta_\epsilon)\tau/(\lambda_N p_r I_X) = 0 \tag{25}$$

$$\mathbf{K}_{N_{SS}} = -D\mathbf{K}_{T_{SS}}^2 - F\mathbf{K}_{T_{SS}} \qquad \mathbf{K}_{P_{SS}} = 0$$

where D is defined in Eqs. (22) and $F = [\Delta(I_X/I - \frac{1}{4})\tau]/\lambda_N$. comparing the first of Eqs. (22) with that of Eqs. (25) shows that, by virtue of the added detuning terms, the latter may possess three real roots. Figure 3 compares the near-resonance theory with 3-degree-of-freedom numerical integrations. It appears that the detuning concept is quite accurate with regard to steady-state oscillations near resonance.

Evaluations of the transient nature of the approximate solution also indicated good comparison with 3-degree-of-freedom calculations, although the accuracy was less than for resonance oscillations. This is felt to be due to the added approximation introduced by the detuning of the rolling velocity. Figure 4 presents $|\alpha|$ as a function of time for the aerodynamics of Fig. 3 at $p/\omega_{N_0} = 1.20$. Slight differences in amplitude and phase occur, although the composite nature of the oscillations is well predicted. Figure 5 shows similar results for a completely nonlinear aerodynamic system: restoring, damping, and Magnus moments. Additional evaluations have been performed yielding similar results.

Recall that in both resonance and near-resonance theories, the rolling velocity was assumed constant. Therefore, the effects of the induced side moment have not been included in previous evaluations. The detuning concept, however, permits the inclusion of a variable roll rate (and roll orientation angle) so as to enable the analysis of nonlinear catastrophic yaw.

Nonliner Catastrophic Yaw

Since the detuning parameter is a measure of the roll rate's proximity to its resonant value, allowing it to vary with time in Eqs. (24) should yield, upon numerical solution, the variations of the amplitude and phase with time in the presence of a variable roll rate. The same is true for variations of γ with time.

Considering first the passage through resonance in the linear case, the variable detuning concept has been found capable of predicting both lag and fractional amplification[23] of the nonrolling trim behind that predicted by classical linear theory. The degree of these effects is dependent on the roll acceleration through the resonance point. Figure 6 presents a simulated roll rate-time history, excluding induced effects, through the resonance region. For linear aerodynamics, Figure 7 shows the angular response resulting from this roll behavior as predicted by both the linear and nonlinear (variable detuning) theories. The former predicts a maximum angle of attack at the resonance point whereas the latter correctly describes the true response (with a small phase shift).

As discussed above, the variable detuning concept also allows the inclusion of $\gamma(t)$ in addition to $p(t)$. These conditions are frequently encountered by missiles susceptible to roll lock-in and catastrophic yaw. To evaluate the nonlinear theory under these circumstances, Figs. 8 and 9 present $p(t)$ and $\gamma(t)$ as produced by an induced roll moment nonlinear in

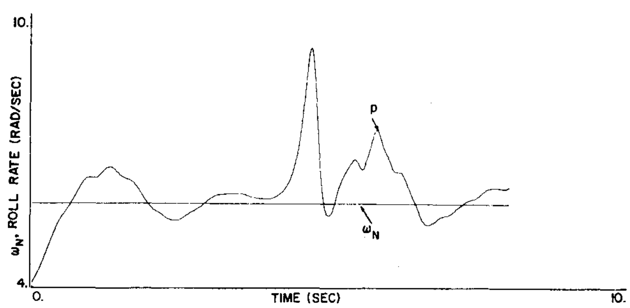

Fig. 8 Roll rate vs time (roll lock-in).

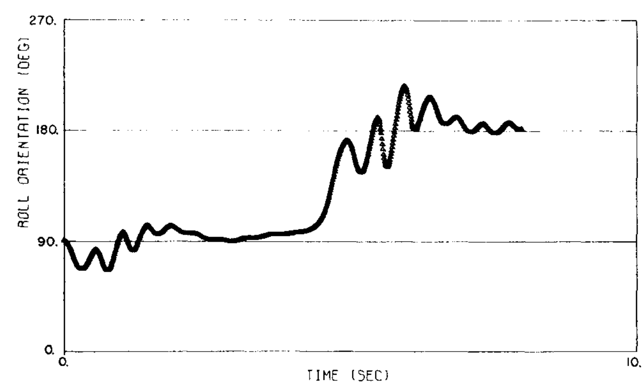

Fig. 9 Roll orientation vs time (roll lock-in).

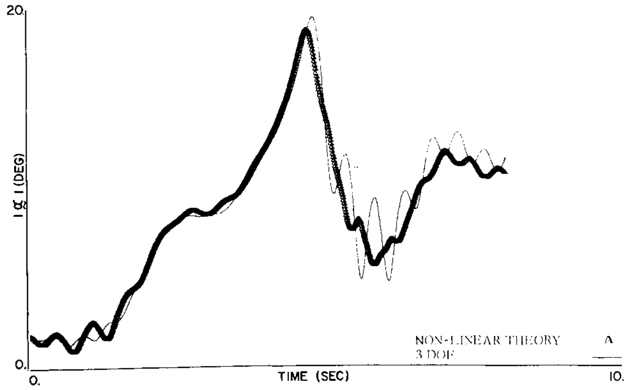

Fig. 10 Nonlinear catastrophic yaw.

angle of attack (with a roll damping moment linear in p). These variations were obtained from 3-degree-of-freedom computations involving an induced side moment nonlinear in angle of attack. Figure 10 presents a comparison of the nonlinear theory and 3-degree-of-freedom computations. The theory is seen to predict the large growth in $|\alpha|$ due to the locked in roll rate and induced side moment. In addition, the reduction in $|\alpha|$ due to the loss of lock-in (as a result of the nonlinearity of the induced roll moment with angle of attack) is also generally predicted, although the degree of ellipticity of the motion as indicated by the theory is somewhat less than could be desired. It is felt that this is a result of the averaging process of the theory being applied during the rapidly changing $\gamma(t)$ of the latter portion of the simulation. It should be noted that the predicted angular response was obtained employing $p(t)$ and $\gamma(t)$ from 3-degree-of-freedom computations. This is necessary since the roll behavior is strongly dependent on $|\alpha|$ during catastrophic yaw. Additional computations have been performed[11] indicating that the variable detuning concept is capable of adequately handling various combinations of nonlinearities in the restoring, damping, Magnus, and induced side moments in the presence of variable roll rate and roll orientation provided that $\dot{p}(t)$ and $\dot{\gamma}(t)$ are not extremely large.

Conclusions

Employing the method of slowly varying parameters, approximate analytic solutions have been presented for the angular motion at and near resonance of finned configurations having nonlinear aerodynamic properties. In addition to these constant rolling velocity analyses, the nonlinear catastrophic yaw problem has been solved approximately, including variations of both roll rate and roll orientation with time. These approximate solutions have been obtained in terms of amplitude-rate equations. Although these first-order differential equations may not be analytically solvable in the most general cases, they do provide the necessary relationship for examining the dynamic stability of each mode (nutation, precession, trim) of oscillation.

Evaluations of the approximate solutions have been performed for several cases by comparing theoretical predictions of the angular response with that obtained from numerical integrations of the equations of motion. Favorable agreement was obtained in all cases.

It was shown that the nonlinear catastrophic yaw solution required $p(t)$ and $\gamma(t)$ as inputs. Efforts should be undertaken to extend this to include the roll equation in the nonlinear solution. The resulting equations would give the capability, without performing lengthy computations, to predict the dynamic stability characteristics of finned configurations experiencing roll lock-in and catastrophic yaw in the presence of nonlinear pitch, yaw, and roll aerodynamic torques.

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